

# Outer space and Automorphisms of free groups

## LECTURE 8

Last time we

1. Found a smaller chain complex which computes  $H^*(\text{Out } F_n)$

2. Defined a Lie algebra of derivations of the free Lie algebra

1.  $C_k = \mathbb{Q}$ -vector space generated by pairs  $(G, \Phi)$

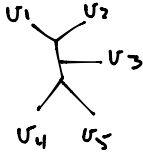
with  $\begin{cases} G \text{ connected, trivalent, rank } n \\ \Phi \subset G \text{ a forest with } k \text{ (ordered) edges} \end{cases}$

modulo anti-symmetry and IHX relations

$$\delta: C_k \longrightarrow C_{k+1} \quad \text{is } \delta = \delta_R + \delta_C$$

$$(G, \Phi) \longmapsto \underbrace{\sum (G, \Phi \cup e)}_{\delta_R} + \underbrace{\sum (G^\alpha, \Phi^\alpha)}_{\delta_C}$$

(sign: new edge is last in ordering)

2.  $\mathfrak{h}$  is generated by spiders  = binary trees, leaves labelled by vectors in  $V = \text{symplectic vector space} / \text{AS, IHX}$

$$[S, S'] = \sum_{\substack{l \in S \\ l' \in S'}} \text{mate } S \text{ and } S' \text{ using } l \text{ and } l'$$

Claim.  $H_*(\mathfrak{h})$  is closely related to  $H^*(\text{OrbFn})$

How do you compute homology of a Lie algebra

(and why is it defined this way?)

Answer: Lie algebra = tang. space to id in a Lie group

= linear approximation to the Lie group.

If the Lie group is compact & simply-connected, it is determined by its Lie algebra, so you should be able

to compute its cohomology from the Lie algebra, too

Lie algebra cohomology was defined to do this

$H^*$  Lie group = deRham cohomology

$C_k$  = Chains are differential forms  $\int dx_1 \wedge \dots \wedge dx_k$

etc.

This motivates defin of Lie algebra (co)homology

$\mathfrak{h}_d$  = Lie algebra  $C_k = \wedge^k \mathfrak{h}_d$

$$\partial : C_k \longrightarrow C_{k-1}$$

$$x_1, \dots, x_k \longmapsto \sum_{i < j} [x_i, x_j] \wedge x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k$$

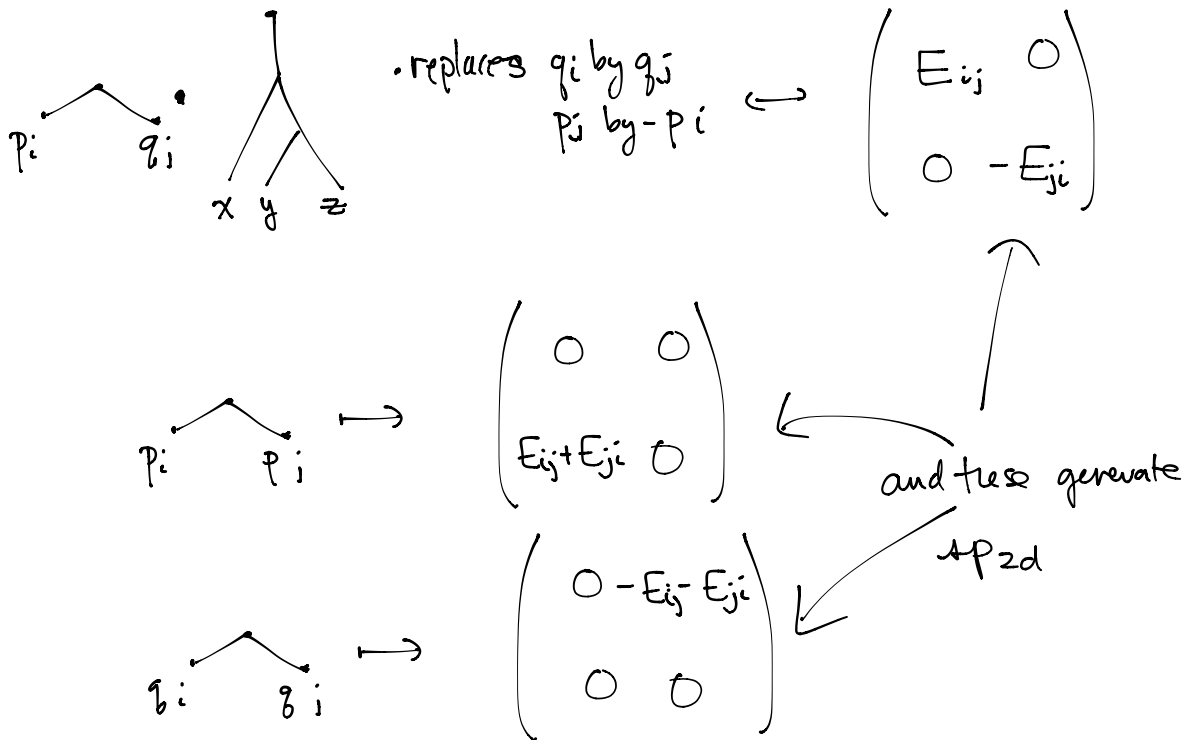
Doesn't look much like our cochain complex for OctFn!

Trick:  $\mathfrak{h}_d$  contains a copy of  $\mathfrak{sp}_d$

= 2-legged spiders

Recall  $\mathfrak{sp}_{2d}$  = matrices  $A$  w/  ${}^t A J + J A = 0$ ,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$



Next note: 2-legged spiders act on  $\mathfrak{h}_d$ , (preserving # of legs, in fact):

$$\Delta_0 \cdot \Delta_1 \wedge \dots \wedge \Delta_k = \sum_{i=1}^k [\Delta_0, \Delta_i] \wedge \Delta_1 \wedge \dots \wedge \widehat{\Delta_i} \wedge \dots \wedge \Delta_k$$

So  $C_k = \wedge^k \mathfrak{h}_d$  is an  $\mathfrak{sp}_{2d}$ -module

Exercise  $\partial$  commutes with the  $\mathfrak{sp}$ -action, so  $(C_k^{\mathfrak{sp}}, \partial)$  is a sub-chain complex ( $C_x^{\mathfrak{sp}}$  = chains killed by  $\mathfrak{sp}$ -action)

**Prop**  $H_* C_k = H_*(C_k^{\text{sp}})$

Proof (deferred) requires fact that  $\text{sp}_{2d}$  is simple.

So we just need to figure out the invariants in  $C_k - \Lambda^k \mathfrak{h}$

Actually, we don't need to figure them out, Weyl already did it!:

$V =$  symplectic vector space

$(V \otimes V)^{\text{sp}}$  is 1-dimensional, gen. by  $\sum_i p_i \otimes q_i - q_i \otimes p_i := \omega$

$(V^{\otimes k})^{\text{sp}} = 0$  unless  $k$  is even,  $k=2l$ . Then  $\underbrace{\omega \otimes \dots \otimes \omega}_l$  is an invariant. Shorthand notation for this:



Any such chord diagram gives another invariant, ed

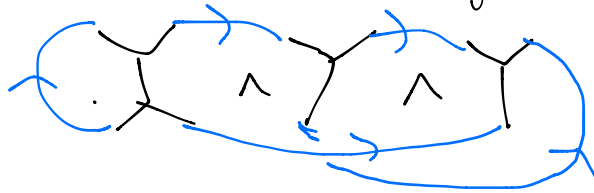


permutes all terms of  $\omega \otimes \omega \otimes \omega$  by  $\sigma = (253)$

Weyl's theorem says these span the entire space of invariants.

This translates to our situation as follows:

Invariants of  $\Lambda^k h \leftrightarrow$  wedges of spiders whose legs are paired.

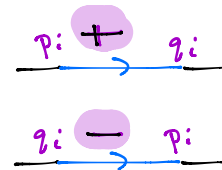


The invariant in  $\Lambda^k h$  is a (large!) sum: a wedge of spiders is in the sum if the labels on the paired edges are a

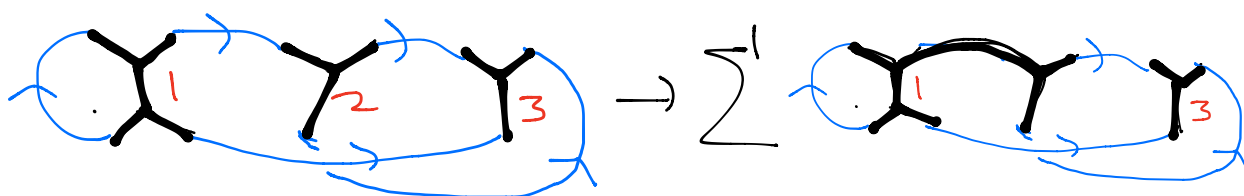
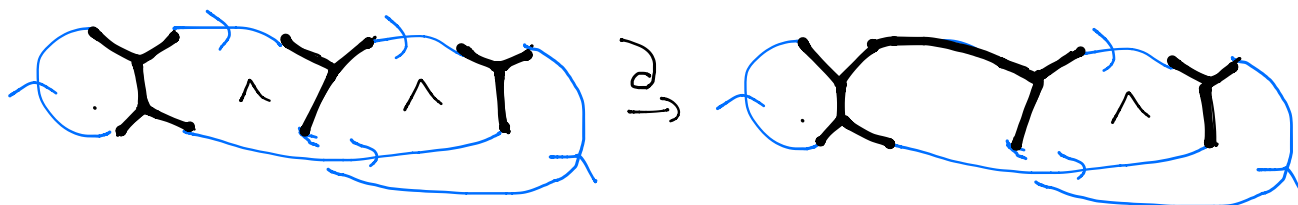
$P_i - Q_i$  pair:  $\xrightarrow{P_i} \xrightarrow{Q_i}$  or  $\xrightarrow{Q_i} \xrightarrow{P_i}$

Sign of the term: each edge gets a sign

sign of term =  $\prod_e \text{sign}(e)$



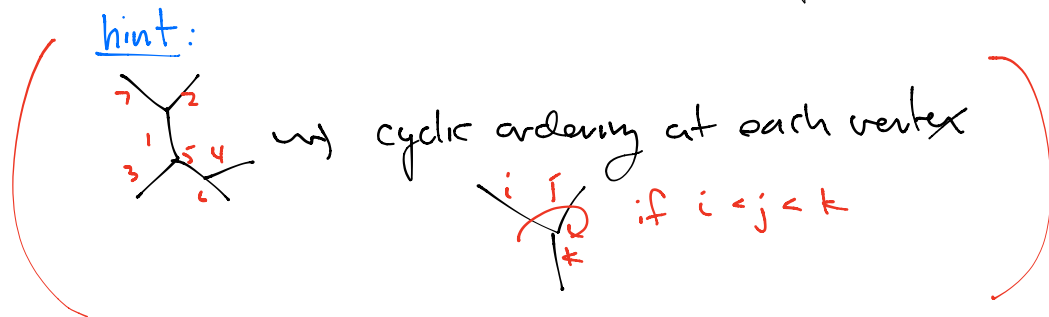
Relationship with our old chain complex:



$$(G, \Phi) \xrightarrow{\partial} \sum (G, \Phi_{ue})$$

**Prop** The orientation on  $(G, \Phi)$  given by ordering the trees in  $\Phi$ , choosing a planar embedding of each tree in  $\Phi$  and orienting the remaining edges is equivalent to that given by ordering the edges of  $\Phi$ .

Proof is not transparent, recipe is complicated  
 basically takes orientation on the space of half-edges of  $G$  and groups the half-edges to produce canonical orientations of both types.



Now we want to compute  $H_\alpha(h_d)$  ... BUT

Actually, this doesn't quite work: we need to take a limit  $h_\infty = \lim_{d \rightarrow \infty} h_d$ ,

ie we allow ourselves an infinite supply of distinct vertex labels.

What is the issue?



We have a map

$$\Phi_d: \mathcal{G}_k \longrightarrow \Lambda^k \mathfrak{h}_d$$

$$(\mathcal{G}, \Phi) \longmapsto \text{sp-invariant}$$

There is also a map

$$\Psi: \Lambda^k \mathfrak{h}_d \longrightarrow \mathcal{G}_k :$$

pair legs of  $S_1 \wedge \dots \wedge S_k$  in all possible ways,  
multiply result by

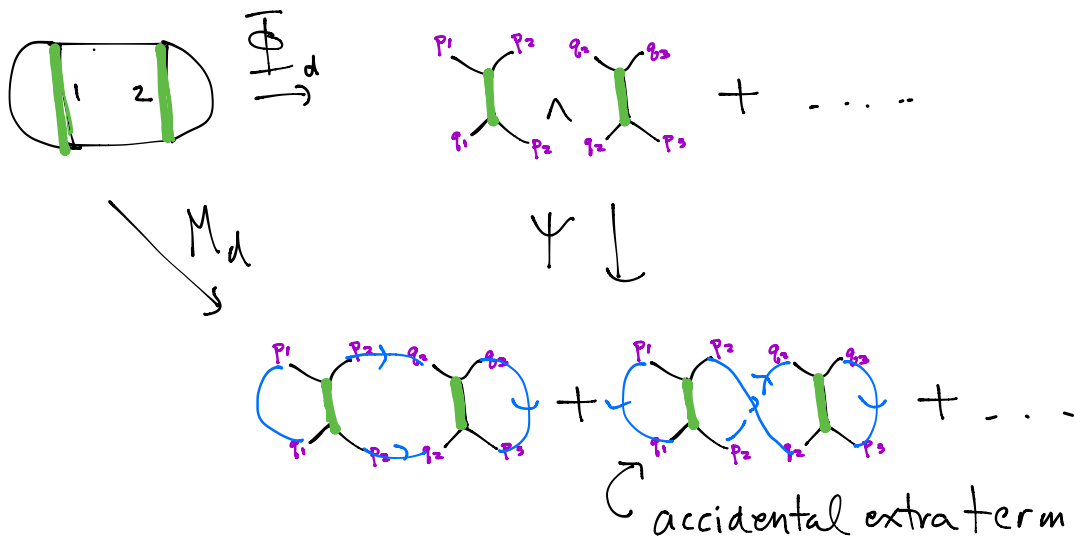
$$\prod_{l_i, l_j \text{ paired}} \langle \sigma_i, \sigma_j \rangle$$

$\sigma_i = \text{label on leg } l_i$

Looks like an inverse to  $\Phi_d$

But  $\Psi \circ \Phi_d = M_d$  is not (a multiple of) Id:

there are accidental pairings



Thm:  $d$  sufficiently large  $\Rightarrow M_d$  is an isomorphism

$$\text{So } \mathcal{G}_d \xleftarrow{\quad} \Lambda^k \mathfrak{h}_d \longrightarrow \mathcal{G}_d$$

(Weyl) image =  $(\Lambda^k \mathfrak{h}_d)^{\text{sp}}$

$$\text{So } H_*(\mathcal{G}_*) = H_*(\Lambda^* \mathfrak{h}_\infty)$$

$$\bigoplus_n H_*(\text{Out } F_n) \oplus H_*(\mathcal{L}P_\infty)$$

Now: let's find some homology!

(actually slightly more natural to look for cohomology...)

Abelianize

$$h \longrightarrow h_{ab} \quad \nwarrow \text{Lie algebra, } [\cdot, \cdot] = 0$$

Get backwards map  $H^*(h_{ab}) \longrightarrow H^*(h)$

Since  $[\cdot, \cdot] = 0$  on  $h_{ab}$ ,  $d = 0 (\Rightarrow \delta = 0)$

$$\Rightarrow H^k(h_{ab}) = \Lambda^k h_{ab} \longrightarrow H^k(h) = \bigoplus H_*^k(\text{Aut } h)$$

Now have to find elements of  $h_{ab}$

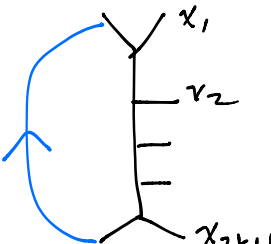
In  $h_{ab}$ , brackets are 0

eg  $\begin{matrix} x & & z \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ y & & w \end{matrix} = \left[ \begin{matrix} x & & y \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ y & & z \end{matrix}, \begin{matrix} z & & y \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ y & & z \end{matrix} \right]$

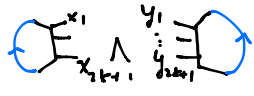


$$\Rightarrow \begin{matrix} x & & z \\ & \diagdown & / \\ & \text{---} & \\ & / & \diagdown \\ y & & w \end{matrix} = 0 \text{ in } h_{ab}$$

So the only tree in  $h_{ab}$  is  $\begin{matrix} x & & y \\ & \diagdown & / \\ & z & \end{matrix} = - \begin{matrix} y & & x \\ & \diagdown & / \\ & z & \end{matrix} = \dots$

So  $h_{ab}$  contains a copy of  $\Lambda^3 V$

$h_{ab}$  also contains 

$$= \sum_{i>1} q^i p^i \begin{matrix} x_1 \\ \vdots \\ x_{2k+1} \end{matrix}$$

So  $\Lambda^2 h \ni$    $\rightarrow$    $+$    $+$   $\dots$

$$\in C_{4k}(\text{Out} F_{2k+2})$$

$$\sum_{\sigma} \text{Diagram} = \mu_k = k^{\text{th}} \text{ Morita class}$$

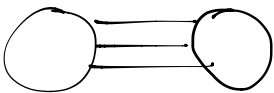
**Conjecture:**  $\mu_k$  represents a non-trivial homology class  
(true for  $k=1, 2, 3$ )

Now that we know where to look, we can see  $M_K$  directly in the quotient of Outer space by  $\text{Out}(F_n)$ , ie in the moduli space of metric graphs  $\mathcal{Q}_n$ .

eg  $\mu_1 \in \mathcal{Q}_4$  is the top-dimensional class of a manifold  $M_4 \subset \mathcal{Q}_4$ .  $M_4$  is the image of a torus  $T^4 = S^1 \times S^1 \times S^1 \times S^1$

Each  $S^1$  is a circle of metric graphs, obtained

by moving the attaching point of a horizontal edge

in  around one of the circles.

There are symmetries in this picture ( $\mathbb{Z}/2 \times \mathbb{Z}/2$ )

so the torus is not embedded in  $\mathcal{Q}_4$ , just

the quotient  $M_4 = T^4 / \mathbb{Z}/2 \times \mathbb{Z}/2$ .

Note  $\dim \mathcal{O}_4 = 3 \cdot 4 - 3 = 9$ , so it is possible that  $M_4$  bounds, i.e. is trivial in homology

In fact  $M_4$  is not a boundary, but we don't have a general proof which works for all  $\mu_k$ .